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The stability of a thermally radiating stratified shear layer

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A linear stability analysis is applied to a stratified, thermally radiating, unbounded shear layer. Temperature disturbances are assumed to be optically thin. Both viscous and inviscid neutral stability boundaries are determined numerically for hyperbolic-tangent mean velocity and potential-temperature profiles. For these profiles, long-wavelength disturbances are completely destabilized (the critical Richardson number $Ri_c \to \infty$ as the wavenumber $k \to 0$) in the inviscid limit. A similar situation is found for the case of discontinuous step-function profiles. However, in contrast to the non-radiating problem, the functional form of the neutral stability boundary is not the same for both the continuous and discontinuous profiles in the limit $k \to 0$. Application of the viscous results to the atmospheres of the earth and Venus yield critical Richardson numbers in excess of $\frac{1}{4}$.

1. Introduction

Radiative heat transfer tends to smooth temperature perturbations in a thermally stratified fluid layer. For an unstably stratified system, the Bénard problem, radiation is stabilizing and leads to increased values of the critical Rayleigh number (see Christophorides & Davis 1970 for discussion). In the case of a stably stratified shear layer, radiative temperature smoothing is a destabilizing mechanism and should lead to increased values of the critical Richardson number.

In the terrestrial atmosphere, turbulence is known to exist for overall Richardson numbers in excess of the non-radiating, inviscid, local value of $\frac{1}{4}$. Recently, difficulties have been encountered in explaining the stability of CO_2 in the upper atmospheres of Venus and Mars (Lewis 1971; Donahue 1971). Sunlight in the upper atmospheres of these planets should dissociate CO_2 into CO and O. It has been postulated that turbulent mixing with relatively large eddy coefficients (see Ingersoll & Leovy 1971) transports dissociation products downwards and CO_2 upwards. At these altitudes, however, one finds stable lapse rates and overall Richardson numbers again in considerable excess of $\frac{1}{4}$. One possible explanation for these phenomena is the occurrence of localized regions of increased shear where the Richardson number drops below $\frac{1}{4}$ (see Maslowe 1972). At the same time, however, radiative destabilization should modify the $\frac{1}{4}$ criterion, whether it is based on a gross Richardson number or on a local value.

A linear stability analysis will be employed in an effort to assess the role of radiation as a destabilizing mechanism. A recent analysis by Miller & Gage

(1972) reported increased critical Richardson numbers for low Prandtl number fluids. However, for non-zero Prandtl number this effect of increased conductivity disappears in the inviscid limit. Townsend (1958) considered the effect of radiation in lessening the buoyant suppression of turbulent motion in a stably stratified fluid. He considered the equations of mean-square turbulent fluctuations and, through assumptions concerning disturbance correlations, was able to determine a relationship between the turbulent intensity, radiative transfer and Richardson number. For a sufficiently large value of a parameter representing the ratio of radiative to convective transfer rates, critical Richardson numbers were found to increase linearly with this parameter. Brutsaert (1972) carried out a similar analysis including the effects of moisture. This type of analysis is independent of Reynolds number, and it is also independent of the role of the inflexion point (of the mean velocity profile) in establishing the instability. A viscous, unstratified, free shear layer is unstable for all Reynolds numbers in the zero wavenumber limit (see Betchov & Szewczyk 1963; Drazin 1961). Since radiation reduces the stabilizing buoyancy effects of stratification, one might expect large critical Richardson numbers, in the small wavenumber limit, for a radiating inflexional shear layer.

The present investigation considers the stability of such a stratified shear layer. In this model temperature disturbances are assumed to be optically thin, and thus the shear-layer depth must be small compared with the photon mean free path length. Hyperbolic-tangent mean velocity and potential-temperature mixing-layer profiles are employed. For these profiles, under the Boussinesq approximation, our viscous equations reduce in the non-radiating limit to those for the Holmboe profiles examined by Maslowe & Thompson (1971). The non-radiating inviscid form of these equations was solved by Holmboe (cf. Drazin & Howard 1966).

In the absence of radiation it is possible to determine the small wavenumber stability characteristics of an unbounded stratified flow by examining the stability of an appropriate model with discontinuous mean profiles (Drazin & Howard 1966; Gage 1972). In line with the above-mentioned anticipation of small wavenumber destabilization, this technique will be examined for the inviscid radiating shear layer, in order to assess the influence of the specific choice of mean profiles on the small wavenumber stability characteristics. Step functions are the appropriate discontinuous profiles.

2. Governing equations

We shall consider the stability of a stratified shear layer with velocity

$$\mathbf{U}(z) = (U(z), 0, 0)$$

and temperature $\overline{T}(z)$, where z is the vertical co-ordinate, positive upwards. The linear Boussinesq equations (Spiegel & Veronis 1960) governing disturbances to this basic state are given by

$$\frac{d\mathbf{u}}{dt} + w\frac{\partial \mathbf{U}}{\partial z} = -\frac{1}{\rho_0}\nabla p - \frac{\mathbf{g}}{T_0}\phi + \nu\nabla^2\mathbf{u}, \tag{2.1a}$$

$$\frac{d\phi}{dt}+w\frac{\partial\theta}{\partial z}=\frac{\kappa}{\rho_0 c_p}\nabla^2\phi-\frac{\nabla\cdot\mathbf{q}}{\rho_0 c_p}, \tag{2.1b}$$

$$\nabla\cdot\mathbf{u}=0, \tag{2.1c}$$

where $$\frac{d}{dt}=\frac{\partial}{\partial t}+U\frac{\partial}{\partial x} \quad\text{and}\quad \frac{\partial\theta}{\partial z}=\frac{\partial\overline{T}}{\partial z}+\frac{g}{c_p}.$$

In this notation $\mathbf{u}=(u,v,w)$, ϕ, p and $\mathbf{q}$ represent perturbations to the velocity, temperature, pressure and radiative heat flux, respectively. The acceleration due to gravity $\mathbf{g}=(0,0,-g)$, a reference density ρ_0, a reference temperature T_0, the kinematic viscosity ν, the coefficient of thermal conductivity κ and the specific heat at constant pressure c_p are all assumed to be constant.

The vertical length scale of the shear layer will be assumed to be small compared with the photon mean free path length. Under this condition it will be further assumed that temperature perturbations are optically thin. In this limit the linearized form of the equation of radiative transfer can be found from Goody (1964, chap. IX) to be given by

$$\nabla\cdot\mathbf{q}=16\alpha_p\sigma\overline{T}^3\phi, \tag{2.2}$$

where α_p is the Planck mean absorption coefficient and σ is the Stefan–Boltzmann constant. It is to be noted that Goody's formulation applies to temperature disturbances of small wavelength, whereas in the current investigation 'optically thin' refers to a disturbance of small vertical dimension. Finally, we shall assume that variations in $\overline{T}$ over the depth of the shear are small compared with T_0 (in line with the Boussinesq approximation), and thus

$$\nabla\cdot\mathbf{q}=16\alpha_p\sigma T_0^3\phi. \tag{2.3}$$

The profiles of the basic state will now be specified by

$$U=\Delta U\tanh(z/L) \quad\text{and}\quad \theta=T_0+\Delta\theta\tanh(z/L).$$

With this choice of velocity and potential-temperature profiles, our final equations, in the absence of radiative transfer, will reduce to the final equations of Maslowe & Thompson (1971; viscous) and Maslowe & Kelly (1971; inviscid), and the eigenvalue problems will be identical. The potential-temperature profile is different from that used by these authors, and this is necessary to compensate for the slightly different form of the Boussinesq approximation employed in the present analysis.

We non-dimensionalize the disturbance equations using the following scaling: L for length, ΔU for velocity, $\rho_0(\Delta U)^2$ for pressure, $\Delta\theta$ for temperature and $L/\Delta U$ for time. Henceforth, all equations will be in non-dimensional form and the notation specified above will now represent non-dimensional quantities. If the equation of radiative transfer is substituted into the energy equation, the Boussinesq equations can be written in non-dimensional form as

$$\frac{d\mathbf{u}}{dt}+w\frac{\partial\mathbf{U}}{\partial z}=-\nabla p+Ri\,\phi\mathbf{k}+\frac{1}{Re}\nabla^2\mathbf{u}, \tag{2.4a}$$

$$\frac{d\phi}{dt}+w\frac{\partial\theta}{\partial z}=\frac{1}{P\,Re}\nabla^2\phi-\frac{16\tau}{Bo}\phi, \tag{2.4b}$$

$$\nabla\cdot\mathbf{u}=0, \tag{2.4c}$$

where $\mathbf{k}$ is a unit vector in the positive z direction. The Reynolds number Re, Richardson number Ri, Prandtl number P, half-layer optical thickness τ and Boltzmann number Bo are given by

$$Re = \frac{\Delta U L}{\nu}, \quad Ri = \frac{g}{T_0}\frac{\Delta\theta}{L}\Big/\left(\frac{\Delta U}{L}\right)^2, \quad P = \nu\Big/\frac{\kappa}{\rho_0 c_p}, \quad \tau = \alpha_p L, \quad Bo = \frac{\rho_0 c_p \Delta U}{\sigma T_0^3}. \tag{2.5}$$

The dimensionless basic state is now given by

$$U = \tanh z \quad \text{and} \quad \partial\theta/\partial z = \mathrm{sech}^2 z.$$

The above system of equations is to be solved subject to the boundary conditions

$$\mathbf{u}, \phi \to 0 \quad \text{as} \quad z \to \pm\infty. \tag{2.6}$$

We shall assume that $\tau < \frac{1}{2}$, in order for the transparent approximation to be valid. This sets an upper bound on L, the scale height of the shear layer.

Normal modes will now be employed; if f is any disturbance quantity then $f(x,y,z,t) = \hat{f}(z)\, e^{i(kx+ly-kct)}$, where $c = c_r + ic_i$. Equations (2.4a–c) become

$$ik(U-c)\,\hat{u} + \hat{w}DU = -ik\hat{p} + (Re)^{-1}(D^2-k^2-l^2)\,\hat{u}, \tag{2.7a}$$

$$ik(U-c)\,\hat{v} = -il\hat{p} + (Re)^{-1}(D^2-k^2-l^2)\,\hat{v}, \tag{2.7b}$$

$$ik(U-c)\,\hat{w} = -D\hat{p} + Ri\,\hat{\phi} + (Re)^{-1}(D^2-k^2-l^2)\,\hat{w}, \tag{2.7c}$$

$$ik(U-c)\,\hat{\phi} + \hat{w}D\theta = (P\,Re)^{-1}(D^2-k^2-l^2)\,\hat{\phi} - G\hat{\phi}, \tag{2.7d}$$

$$D\hat{w} + ik\hat{u} + il\hat{v} = 0, \tag{2.7e}$$

where $D = d/dz$ and $G = 16\tau/Bo$ is the ratio of radiative to convective heat transfer of the undisturbed flow.

This system of equations can be reduced to an equivalent two-dimensional system ($\hat{v} = l = 0$) by means of Squire's transformation (see Gage & Reid 1968 for details). If $\bar{Ri}$, $\bar{Re}$, $\bar{G}$, $\bar{P}$ and $\bar{k}$ represent the dimensionless parameters of this equivalent two-dimensional system, they are related to those of the complete three-dimensional system by

$$\bar{k}^2 = k^2 + l^2, \quad \bar{p} = p, \quad \bar{Re} = (k/\bar{k})\,Re, \quad \bar{Ri} = (\bar{k}/k)^2\,Ri, \quad \bar{G} = (\bar{k}/k)\,G. \tag{2.8}$$

In the absence of radiation ($G = 0$), the result of Gage & Reid (1968) follows. This states that two-dimensional disturbances are the most unstable, since any three-dimensional disturbance is equivalent to a two-dimensional disturbance at lower Reynolds number and higher Richardson number. This is not the case if radiation is included, because the two-dimensional problem is now at a larger value of G than the three-dimensional problem.

From a knowledge of the two-dimensional stability boundary $\bar{Ri}\,(\bar{k}, \bar{Re}, \bar{G})$, the stability boundary for an arbitrary three-dimensional disturbance can be constructed using (2.8). If for fixed $\bar{Re}$, $\bar{Ri}$ increases less rapidly than $\bar{G}^2$ (as $\bar{G}$ or $\bar{G}^{\frac{3}{2}}$, for example), then two-dimensional disturbances will have a larger Richardson number and a smaller Reynolds number than any three-dimensional disturbance with the same G. In the present problem this is found to be the case, and thus two-dimensional disturbances are the most unstable.

From here on we shall consider only two-dimensional disturbances (the bar will be dropped). Since we are interested in the neutral stability boundary we shall assume that the growth rate $c_i = 0$. Furthermore, owing to the antisymmetry property of the basic profiles, we can assume that $c_r = 0$ (see Maslowe & Thompson 1971) and that $\hat{\phi}(z) = \hat{\phi}^*(-z)$ and $\hat{u}(z) = \hat{u}^*(-z)$, where $\hat{u}^*$ represents the complex conjugate of $\hat{u}$. Under these conditions, the two-dimensional form of equations (2.7 *a–e*) can be combined into the following two equations:

$$(D^2-k^2)^2\hat{w} = ik\,Re\{U(D^2-k^2)\hat{w} - U''\hat{w} - ik\,Ri\,\hat{\phi}\}, \qquad (2.9a)$$

$$(D^2-k^2)\hat{\phi} = P\,Re\{ikU\hat{\phi} + \theta'\hat{w} + G\hat{\phi}\}. \qquad (2.9b)$$

These equations are to be solved subject to

$$\hat{w}, \hat{\phi} \to 0 \quad \text{as} \quad z \to \pm\infty. \qquad (2.10)$$

For the case of no radiative transfer, the above system (including the functional forms of U and θ') is equivalent to that investigated by previous authors.

3. Inviscid stability boundary

In the inviscid limit ($Re, k\,Re \to \infty$), equations (2.9 *a, b*) reduce to the form

$$U(D^2-k^2)\hat{w} - U''\hat{w} - ik\,Ri\,\hat{\phi} = 0, \qquad (3.1a)$$

$$ikU\hat{\phi} + G\hat{\phi} + \theta'\hat{w} = 0. \qquad (3.1b)$$

These two equations can be combined to yield

$$(D^2-k^2)\hat{w} - \frac{U''}{U}\hat{w} + \frac{Ri\,\theta'}{U(U-iG/k)}\hat{w} = 0, \qquad (3.2)$$

which is to be solved subject to

$$\hat{w} \to 0 \quad \text{as} \quad z \to \pm\infty. \qquad (3.3)$$

This equation is singular at $z = 0$. As $z \to 0$, $U \sim z$ and the singularity of the equation $\sim z^{-1}$ for any non-zero G. In contrast, the singularity of the non-radiating problem ($G = 0$) $\sim z^{-2}$. In this case the neutral stability boundary is given by $Ri = k(1-k)$ (cf. Drazin & Howard 1966). It should be noted also that G appears only in the form G/k, and one might expect the effect of radiation to be enhanced in the long wavelength limit $k \to 0$.

The eigenvalue problem represented by (3.2) and (3.3) is to find the stability boundary $Ri(k, G)$. The procedure employed is as follows. We approximate (see Gage 1972) U and θ by $U = \theta = \tanh z$ for $z \leqslant 3$ and $U = \theta = 1$ for $z > 3$. Two linearly independent solutions of (3.2), which remain finite and are valid for $z \to 0$, are found (see appendix A). For each of these solutions the values of the function and its first derivative at $z = \epsilon \ll 1$ ($\epsilon > 0$) are used as starting values to integrate (3.2) numerically to $z = 3$. Following Gage (1972), the values of $\hat{w}$ and $\hat{w}'$ across the singularity at z slightly greater than 3 ($z = 3+$) are found (see appendix B). The solutions of (3.2) at $z = 3+$ are $\hat{w} \sim e^{kz}$, e^{-kz} (since $U = 1$, $\theta' = 0$ there). The correct eigensolution is the decaying solution, and thus at $z = 3+$ the correct boundary condition is given by

$$\hat{w}' + k\hat{w} = 0. \qquad (3.4)$$

It should be noted that an alternative exists to the approximation of U and θ by a constant for $z > 3$. Without this approximation (3.4) maybe applied directly, as long as in (3.2)

$$\left|-\frac{U''}{U}+\frac{Ri\,\theta'}{U(U-iG/k)}\right| \ll k^2.$$

For $z \rightarrow \infty$, U''/U, $\theta'/U \sim e^{-2z}$, and (3.4) may be applied at some $z = \bar{z}$ if $e^{-2\bar{z}} \ll k^2$. Thus, for small k, equation (3.2) must be numerically integrated to a considerable distance before the boundary condition (3.4) is applied. For this reason the method of Gage is used and the equations need to be integrated only to some fixed distance independent of the value of k.

If $\hat{w}$ is the actual eigenfunction to be found, then

$$\hat{w} = \hat{w}_1 + A\hat{w}_2, \tag{3.5}$$

where A is a real constant and $\hat{w}_1$ and $\hat{w}_2$ are the two linearly independent solutions started from $z = \epsilon$. For a given value of Ri we apply (3.4) to (3.5) at $z = 3+$. Since the $\hat{w}$'s are complex, we have two equations for the unknown A. If A_1 and A_2 are the two values for A (which are different, in general, unless Ri is the correct eigenvalue), we increase Ri until $DA = A_1 - A_2$ changes sign, and then use Newton's method to find the crossover, where $A_1 = A_2$ and Ri is the correct eigenvalue of (3.2) and (3.3). For this value, the solution $\hat{w} \rightarrow 0$ as $z \rightarrow -\infty$ owing to the symmetries of (3.2) and the starting solutions found in appendix A.

The stability boundaries for $G = 0{\cdot}1$ and $G = 0{\cdot}5$ are given in figure 1, along with the non-radiating $G = 0$ stability boundary. The surprising result is that $Ri \rightarrow \infty$ as $k \rightarrow 0$. A non-uniform limit exists in the sense that

$$\lim_{G\rightarrow 0} Ri(G,k) \neq Ri(0,k)$$

over all k. This non-uniformity will be cleared up with the inclusion of viscosity, as will be seen below. For any finite Reynolds number the neutral stability curves will return to zero as $k \rightarrow 0$. The inviscid results are only approximations to the viscous problem in the double limit Re, $k\,Re \rightarrow \infty$, and in this case are not valid as $k \rightarrow 0$.

Further inviscid information may be obtained from examination of (3.2). It would appear that as G increases and G/k becomes $\gg 1$, we could neglect U with respect to iG/k in the denominator of the last term in (3.2). In this case Ri appears in combination with G as Ri/G and this becomes the new eigenvalue as a function of k. Thus, for G large enough, Ri becomes a linear function of G for fixed k. This behaviour is illustrated in figure 2, where we have plotted Ri/G *vs.* k for various values of G. The limiting curve marked $G = \infty$ is the solution of (3.2) with the last term approximated by $(i\,Ri\,k\theta'/G)\,\hat{w}$. The approximation can be seen (from figure 2) to be quite accurate for $G \geqslant 5$, even though for $G = 5$, $G/k \leqslant 10$ for some k in the interval 0–1. The approximation becomes valid for any G as $k \rightarrow 0$, and in this case it is found numerically that $Ri/G \sim k^{-\frac{1}{2}}$.

It should be recalled that temperature disturbances are assumed to be optically thin, although instability is predicted at small wavenumber where disturbance horizontal length scales are considerably larger than the shear layer scale height

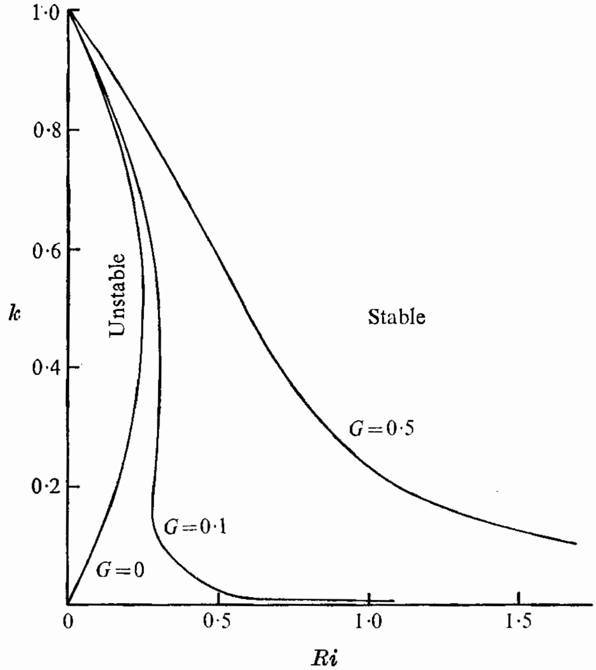

FIGURE 1. Inviscid neutral stability boundary.

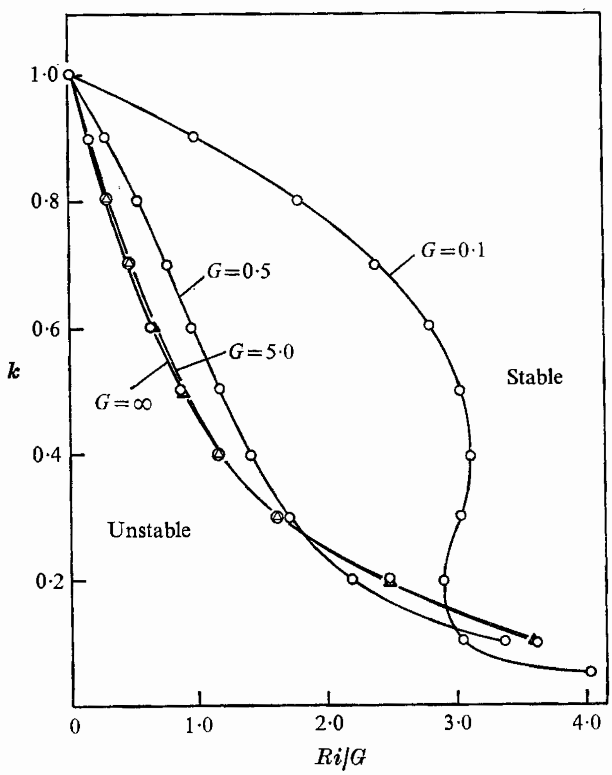

FIGURE 2. Inviscid *Ri/G* boundary.

L. The vertical length scale of the velocity disturbance $\hat{w}$ also extends considerably beyond L (since $e^{-2z} \ll k^2$ before $\hat{w}$ decays exponentially) for small k. Fortunately the temperature disturbance ϕ is found to remain relatively confined to the shear layer. This can be seen from (3.1*b*), since $\hat{\phi} \to 0$ as $\theta' \to 0$. The same result is found in the viscous case, and the transparent approximation will be assumed valid even as $k \to 0$.

In summary, no critical Richardson number exists for the inviscid problem, with even the smallest amount of radiative transfer being allowed. For a given value of G and any Ri, one can always find a wavenumber k small enough such that an infinitesimal disturbance will grow exponentially in time.

4. Viscous stability boundary

The viscous problem consists of finding the stability boundary $Ri(k, P, Re, G)$, such that (2.9*a*, *b*) and (2.10) are satisfied. The method of solution is as follows.

We approximate U and θ by $\tanh z$ for $|z| < 3$, -1 for $z \leqslant -3$ and $+1$ for $z \geqslant 3$. Owing to the symmetry of (2.9*a*, *b*), an eigensolution exists such that $\hat{\phi}^*(z), \hat{w}^*(z) = \hat{\phi}(-z), \hat{w}(-z)$. This leads to the boundary conditions

$$\left.\begin{aligned} \mathrm{Re}(\hat{w}', \hat{w}''', \hat{\phi}') &= 0 \\ \mathrm{Im}(\hat{w}, \hat{w}'', \hat{\phi}) &= 0 \end{aligned}\right\} \quad \text{at} \quad z = 0. \tag{4.1}$$

For $z \leqslant -3$, the decaying solutions are given by

$$\hat{w}, \hat{\phi} \sim e^{\lambda_i z} \quad (i = 1, 3), \tag{4.2}$$

where

$$\lambda_1 = k, \quad \lambda_2 = k(1 - i\,Re/k)^{\frac{1}{2}}, \quad \lambda_3 = k\{1 + GP\,Re/k^2 - iP\,Re/k\}^{\frac{1}{2}}. \tag{4.3}$$

For a given Ri we integrate (2.9*a*, *b*) from $z = -3$ to $z = 0$, employing each of the decaying solutions (4.2) with (4.3). Jump conditions derived in appendix A are used to find the correct values of $\hat{w}$, $\hat{\phi}$ and their derivatives across the discontinuity at $z = -3$. If for $i = 1, 3$, $F_i = (\hat{w}_i, \hat{\phi}_i)$ represents these three solutions, then the actual eigenfunctions $F = (\hat{w}, \hat{\phi})$, which satisfy (4.1) as well as (4.2) with (4.3), can be represented by the linear combination

$$F = (1 + iA)\,F_1 + (B + iC)\,F_2 + (D + iF)\,F_3, \tag{4.4}$$

where A, B, C, D and E are real constants. The F_i are complex, and F has been normalized so that the real part of the complex constant multiplying F_1 is unity. If we apply (4.1) to (4.4) we have six equations for the five unknown constants. Solving two different sets of five equations will yield, in general, different values of the unknowns. If δA, δB, δC, δD and δE represent differences between these values, we increase Ri and repeat the procedure until all these change sign between two values of Ri. Newton's method is used to find the correct eigenvalue Ri where all the differences are zero (meaning that both sets of equations yield the same initial conditions for F).

Results for $G = 0{\cdot}1$ (P is taken to be $0{\cdot}72$ for all viscous calculations) and $Re = 10$, 90 and 500 are given in figure 3. It can be seen that, as Re is increased, the neutral stability curves approach the inviscid results, except in the limit

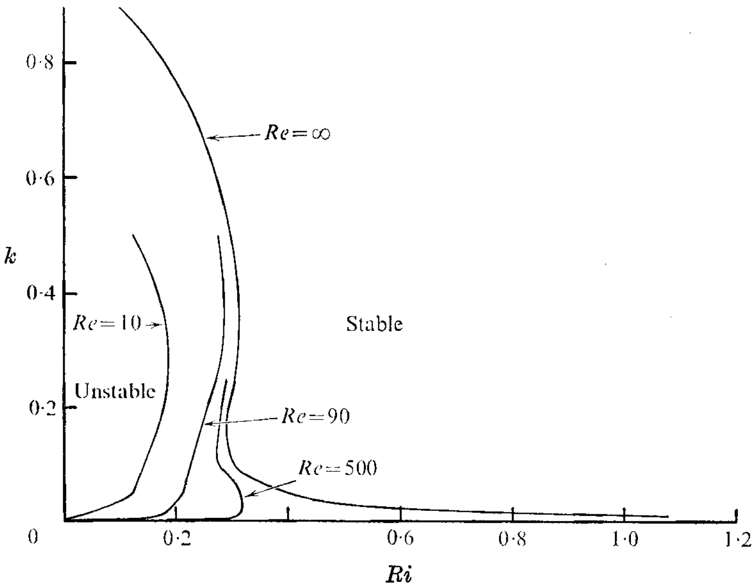

FIGURE 3. Viscous neutral stability boundary for $G = 0{\cdot}1$.

$k \to 0$, where Ri always returns to zero. Thus, in contrast to the inviscid problem, a critical Richardson number (maximum Ri over all k for fixed G and Re) exists; however, the critical Richardson number goes to infinity as $Re \to \infty$ and $k \to 0$.

Employing the numerical scheme described above, it becomes exceedingly difficult to obtain stability curves for large G ($G > 1$) and Re ($Re > 500$), because the growing exponential solutions given by $e^{-\lambda_i z}$ then become large and lead to numerical instabilities. A technique similar to that employed in the inviscid problem for large G/k can be employed here.

Let $\overline{\phi} = \hat{\phi}/G$. Equations (2.9$a$, b) may now be written as

$$(D^2-k^2)^2\hat{w} = ik\,Re\left\{U(D^2-k^2)\hat{w} - U''\hat{w} - \frac{ik\,Ri}{G}\overline{\phi}\right\}, \tag{4.5a}$$

$$(D^2-k^2)\overline{\phi} = GP\,Re\left\{\frac{ikU\overline{\phi}}{G} + \overline{\phi} + \theta'\hat{w}\right\}. \tag{4.5b}$$

In the double limit G/k, $GP\,Re \to \infty$ equations (4.5a, b) can be combined to give

$$(D^2-k^2)^2\hat{w} = ik\,Re\left\{U(D^2-k^2)\hat{w} - U''\hat{w} + \frac{ik\,Ri}{G}\theta'\hat{w}\right\}, \tag{4.6}$$

which is to be solved subject to $\hat{w} \to 0$ as $z \to \pm\infty$.

As in the inviscid case, Ri/G becomes the new eigenvalue. Thus, for sufficiently large G/k and Re, such that (4.6) is a valid approximation to (2.9a, b), Ri is a linear function of G for fixed k and Re. Calculated values of Ri/G (employing various values of Re, k and G) for the full system are found to be within a few per cent of the solution of the approximate system for $G/k = 10$ and for all $Re > 10$. When G/k is increased, the two solutions converge rapidly.

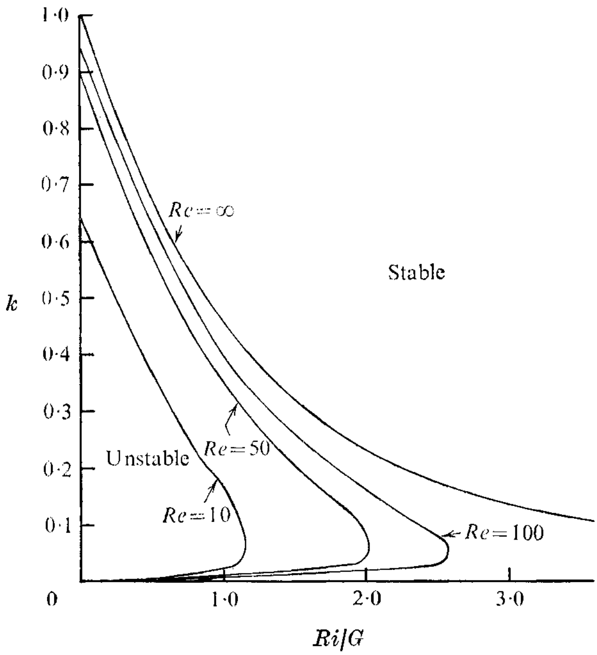

FIGURE 4. Viscous Ri/G boundary in large G/k limit.

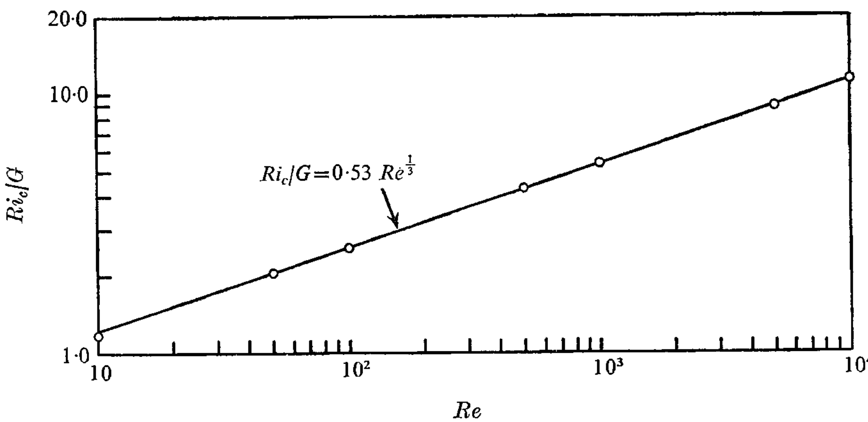

FIGURE 5. Ri_c/G *vs.* Re in large G/k limit.

The approximate equation is computationally less difficult to solve than is the full system, for two reasons. First, the equation is now of fourth order and only two linearly independent solutions have to be considered. Second, the growing exponentials are $\exp[-kz]$ and $\exp[-k\{1+i\,Re/k\}^{\frac{1}{2}}z]$, and these do not grow appreciably large for small k, where the critical Richardson numbers are located. A numerical scheme similar to that employed in solving the complete viscous system is developed for the approximate system.

The stability boundary for such large G/k is shown in figure 4 (along with the inviscid, large G/k boundary) for $Re = 10$, 50 and 100. Additional calculations

Re	k	Ri/G
10	0·05	1·16
50	0·05	2·01
100	0·05	2·55
500	0·03	4·26
1000	0·02	5·31
5000	0·01	8·89
10000	0·006	11·1

TABLE 1. Critical value of Ri/G in the large G/k limit.

are made in the neighbourhood of the critical Richardson numbers for Reynolds numbers up to 10000. The results are given in table 1 and are plotted on a log–log scale in figure 5. Critical values of Ri/G are accurate to within ± 3 in the third significant digit. The corresponding critical values of k are accurate only to ± 1 in the first significant digit, owing to the relative flatness of the neutral stability curves in the neighbourhood of the critical values. From figure 5 it can be determined that the critical Richardson number Ri_c is given by

$$Ri_c = 0{\cdot}53 G\, Re^{\frac{1}{4}}. \tag{4.7}$$

It is to be remembered that (4.7) is an accurate solution for the complete viscous problem only for sufficiently large G/k (say $G/k > 10$). It will be accurate for relatively small G as long as Re is large enough, since the critical value of k decreases with increasing Re.

5. Inviscid step-function profiles

For an inviscid non-radiating stratified shear layer the two limits $k \to 0$ and $L \to 0$ yield identical eigenvalue problems, provided that Ri/k remains fixed (Drazin & Howard 1966). This fact may be employed to determine the stability characteristics of large-scale disturbances ($k \to 0$) for any stratified shear layer by examining the appropriate limiting discontinuous velocity and temperature profiles. In the present case the correct discontinuous profiles in the limit $L \to 0$ are step functions, and in non-dimensional form

$$U = \theta = z/|z|. \tag{5.1}$$

This technique does not usually yield information about the critical Richardson number, since it is valid only for small k and not in the region of maximum Ri.

For the inviscid radiating shear layer the limit $k \to 0$ is considerably more interesting, since $Ri \to \infty$ in this limit. Unfortunately, the arguments leading to the equivalence between the stability boundary of any shear layer and that of the step function do not carry through for the radiating problem, because in this case Ri/k is not fixed, but tends to infinity as $k \to 0$. Physically, one would expect any shear layer to appear as a step function in the limit of large wavelength disturbances. Also it is interesting to determine whether the non-uniformity in $\lim_{G\to 0} Ri(k, G)$ is merely connected with the specific choice of velocity and temperature profiles we have made, or is representative of the transparent radiative model for any stratified inflexional shear layer. For these reasons we shall examine the neutral stability boundary for the inviscid, radiating, step-function profiles.

With U and θ given by (5.1), the decaying solution of (3.2) is given by

$$w = \begin{cases} A\,e^{-kz} & (z > 0), \\ B\,e^{kz} & (z < 0), \end{cases} \tag{5.2}$$

where A and B are constants. The matching conditions across the discontinuity at $z = 0$ already have been derived (equations (B 3, 4) of appendix B). Applying (B 4) on both sides of the discontinuity yields $A = -B$. Now applying (B 3) across the discontinuity, we find

$$2k = Ri\left[2+\frac{iG}{k}\left\{\ln\left(1-\frac{iG}{k}\right)-\ln\left(-1-\frac{iG}{k}\right)\right\}\right]. \tag{5.3}$$

We transform the arguments of the ln terms to exponential form and solve for Ri, giving

$$Ri = \frac{k}{1-(G/2k)(\theta_1-\theta_2)}, \tag{5.4}$$

where $\quad \theta_1 = \tan^{-1}(-G/k), \quad \theta_2 = \tan^{-1}(G/k), \quad -\pi \leqslant \theta_1, \theta_2 \leqslant 0.$

For fixed k, $Ri \to k$ as $G \to 0$. In the limit $k \to 0$ (for fixed G) we may write

$$\theta_1 = -\tfrac{1}{2}\pi + k/G - \tfrac{1}{3}k^3/G^3 + \ldots,$$
$$\theta_2 = -\tfrac{1}{2}\pi - k/G + \tfrac{1}{3}k^3/G^3 - \ldots.$$

Substituting these expressions into (5.4) we find

$$Ri \to 3G^2/k \tag{5.5}$$

as $k \to 0$.

Thus the neutral-stability boundary $Ri\,(k, G)$ for the step-function profiles also exhibits a non-uniformity in the limits k, $G \to 0$. However, the functional form of Ri for $k \to 0$ is not the same in this case as it is for the hyperbolic-tangent profiles, where it was found that $Ri \sim Gk^{-\frac{1}{2}}$. It would appear that the transparent radiation model completely destabilizes a stratified inflexional shear flow, though the exact functional form of this small wavenumber instability is dependent upon the exact profiles studied.

6. Discussion

The results of § 4 will now be applied to several specific examples. We shall consider the upper atmosphere of Venus (100 km) and both the lower and upper atmosphere of the earth. Following Goody (1964), the contributions to the Planck mean absorption coefficient due to water vapour and CO_2 will be given by

$$\alpha_p = (96\rho_{CO_2} + 203\rho_{H_2O})\,\text{cm}^{-1}, \tag{6.1}$$

where the densities are in g cm^{-3}.

For Venus, we shall assume that the atmosphere is entirely CO_2, and thus $\alpha_p = 96\rho$, where ρ is the atmospheric density. If $\Delta U = SL$, where S is the vertical velocity shear (in s^{-1}), then the transparent radiative parameter G is given by

$$G = 1536\sigma T_0^3/c_p S.$$

At 100 km we will take as typical values $T_0 = 150$ °K, $\rho = 7{\cdot}3 \times 10^{-8}$ gm cm^{-3} and $S = 10^{-2}$ s^{-1}. This large shear has been chosen to be representative of the four-day Venus circulation (Gold & Soter 1971). For these values we find that $G = 3{\cdot}5$ and the photon mean free path length $\lambda_p = \alpha_p^{-1} = 1{\cdot}4$ km. The Reynolds number is given by $Re = \rho SL^2/\nu$, and for a 400 m layer ($L = 200$ m) $Re = 3700$. From table 1, the critical wavenumber for $Re = 5000$ is approximately 0·01, and thus G/k is large enough so that (4.7) should be an accurate approximation for the critical Richardson number. On substituting $G = 3{\cdot}5$ and $Re = 3700$ into this equation, we find that $Ri_c = 28$. For lower levels in the Venus atmosphere, G will be greater (owing to larger T_0 and smaller S values). At the same time, λ_p will decrease (being 10 m at 80 km), and consequently only relatively thin layers may be considered transparent.

If, at the 100 km level on Venus, we take the average temperature gradient to be -1 °K km^{-1} and the adiabatic lapse rate to be 9 °K km^{-1}, we have a potential-temperature gradient of 8 °K km^{-1}. For the previously given values of T_0 and $S = \Delta U/L$, we find the actual Richardson number to be 4·5 at 100 km. This is considerably greater than the value $\frac{1}{4}$, though it is less than the critical value of 28 found above. On Mars the same qualitative situation (of Richardson numbers greater than $\frac{1}{4}$ though less than the radiative critical value) exists. This radiative shear destabilization, in an otherwise stable atmosphere, could provide the turbulence and relatively large mixing rates necessary to explain the occurrence of CO_2 as a major constituent in these upper atmospheres.

For the lower troposphere of the earth, we shall take

$$T = 300\ \text{°K}, \quad \rho = 1{\cdot}3 \times 10^{-3}\ \text{g cm}^{-3}, \quad S = 2 \times 10^{-3}\ \text{s}^{-1}.$$

With 300 p.p.m. CO_2 and water vapour at 10% relative humidity (a partial pressure of 3 mbar H_2O), we find $\lambda_p = 34$ m and $G = 0{\cdot}32$. We shall assume that a shear layer is transparent if $2L/\lambda_p < 1$, and this will be satisfied by a 20 m shear layer ($L = 10$ m) with $Re = 15000$. Again assuming the large G/k results to be valid, (4.7) yields a critical Richardson number of 4·2. A higher relative humidity will yield a considerably larger G, though the photon mean free path length becomes small and again only relatively thin layers are transparent (i.e. for 100% relative humidity $G = 4{\cdot}3$ and $\lambda_p = 2{\cdot}6$ m).

A similar calculation at the 20 km level, assuming

$$T = 220\ \text{°K}, \quad \rho = 8{\cdot}9 \times 10^{-5}\ \text{g cm}^{-3}, \quad S = 2 \times 10^{-3}\ \text{s}^{-1},$$

300 p.p.m. CO_2 and no H_2O, yields $\lambda_p = 2{\cdot}5$ km and $G = 0{\cdot}024$. For a 340 m deep shear ($L = 120$ m) $Re = 180000$. This Reynolds number is larger than that found in the lower troposphere, owing to the deeper shear layer being considered. If we again assume that the critical wavenumber is small enough so that (4.7) is valid, we find $Ri_c = 0{\cdot}72$. Radiative effects decrease at higher elevations, owing to the decreased density of absorbing CO_2 molecules.

These examples indicate that the simple model of transparent radiative heat transfer in a stratified shear layer can raise critical Richardson numbers above the value $\frac{1}{4}$ for situations of practical interest. In particular, CO_2 atmospheres such as those of Venus or Mars may not be nearly as stable as one might have previously suspected.

Though the present results differ considerably from those of Townsend, several similarities can be noted. For $G > \frac{1}{16}$ (using the present notation) Townsend found that the critical Richardson number $Ri_c \simeq 2G$. Our results, in the inviscid limit, have $Ri \sim G$ for fixed k provided that G is large enough. If we momentarily neglect the low wavenumber radiating instability and consider the critical Richardson number to be located at $k = 0{\cdot}5$ (the non-radiating critical wavelength), we find (from figure 2) that $Ri_c \simeq 0{\cdot}9G$.

Unfortunately, this small wavenumber instability cannot be shown to be representative of all shear layers with differing velocity and potential temperature profiles (see § 5), as can be done in the case of the non-radiating shear layer. However, the fact that $Ri \to \infty$ as $k \to 0$ for the inviscid step-function profiles, as well as for the hyperbolic-tangent profiles, tends to confirm the presence of this large wavelength instability as a real feature of any transparent inflexional shear layer, not merely a result of the specific profiles chosen in this investigation.

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Appendix A. Inviscid solutions at singularity

In the limit $z \to 0$, we have $U \to z$, $U'' \to -2z$ and $\theta' \to 1$. Therefore, in this limit (3.2) may be written as

$$\hat{w}'' + \left(2 + \frac{Ri\,k^2}{G^2} - k^2 + \frac{i\,Ri\,k}{Gz}\right)\hat{w} = 0. \tag{A 1}$$

For $z \to 0$, the last term in the parentheses should be the largest, and in this case

$$\hat{w}'' + \frac{i\,Ri\,k}{G}\frac{\hat{w}}{z} = 0. \tag{A 2}$$

Two linearly independent solutions of this equation which remain bounded as $z \to 0$ can be given in terms of the Bessel functions J_1 and Y_1 as

$$\hat{w} = tJ_1(t),\ tY_1(t), \tag{A 3}$$

where $t = (4i\,Ri\,kz/G)^{\frac{1}{2}}$. However, for small k the numerical scheme must start at a sufficiently small value of z, such that $Ri\,k/Gz \gg 2$ and (A 2) is a valid approximation of (A 1). Owing to the infinite slope at the origin of the second of the two solutions (A 3), numerical difficulties arise when starting the integration extremely close to the origin. This problem is overcome by using a Frobenius solution to solve (A 1). The resulting solution will be valid over a larger range of z in the small k limit.

Let $A = k^2 - 2 - Ri\,k^2/G^2$, $B = Ri\,k/G$ and $x = iz$. Then (A 1) may be written as

$$\hat{w}'' + A\hat{w} + (B/x)\,\hat{w} = 0. \tag{A 4}$$

Following the usual procedure, we look for a solution of the form

$$\hat{w} = \sum_{l=0}^{\infty} a_l x^{l+c}. \tag{A 5}$$

On substituting this solution into (A 4) we find the recursion relation

$$a_{l+1}(l+c+1)(l+c) + Aa_{l-1} + Ba_l = 0. \tag{A 6}$$

Putting $l = -1$, we have the indicial equation

$$a_0 c(c-1) = 0. \tag{A 7}$$

Since the two roots ($c = 0, 1$) of (A 7) differ by an integer, (A 5) yields only one linearly independent solution. For $c = 0$ we must have $a_0 = 0$ for the a_l (determined by (A 6)) to be finite. A second linearly independent solution can be shown to be

$$\hat{w} = \sum_{l=0}^{\infty} a_l'|_{c=0} x^l + \ln(x) \sum_{l=0}^{\infty} a_l x^l, \tag{A 8}$$

where $a_l'|_{c=0}$ is the value of the derivative of a_l (from (A 6)) with respect to c, evaluated at $c = 0$.

The first four terms of the series (A 5) and (A 8) are used to determine the starting values of $\hat{w}$ and $\hat{w}'$ at $z = \epsilon \ll 1$. The values of the first four coefficients a_l and $a_l'|_{c=0}$ are

$$\begin{aligned}
a_0 &= 0, & a_0' &= 1,\\
a_1 &= -B, & a_1' &= B,\\
a_2 &= \tfrac{1}{2}B^2, & a_2' &= -\tfrac{1}{2}[A + \tfrac{5}{2}B^2],\\
a_3 &= \tfrac{1}{6}[AB - \tfrac{1}{2}B^3], & a_3' &= -\tfrac{2}{9}[AB + \tfrac{5}{4}B^3],\\
a_4 &= -\tfrac{1}{24}[\tfrac{4}{3}AB^2 - \tfrac{1}{6}B^4], & a_4' &= \tfrac{1}{24}[A^2 + \tfrac{33}{72}B^4 + \tfrac{25}{9}AB^2].
\end{aligned}$$

Letting $A = 0$, the solutions (A 5) and (A 8) are in fact the series representation of (A 3).

Appendix B. Matching conditions across basic profile discontinuities

Inviscid problem

After being multiplied by U, equation (3.2) may be written as

$$(U\hat{w}' - U'\hat{w})' - k^2 U\hat{w} + \frac{Ri\, U\theta'\hat{w}}{U(U - iG/k)} = 0. \tag{B 1}$$

Since $\theta' = U'$, the last term in (B 1) may be written as

$$Ri\left\{\frac{\hat{w}}{U}\left(U + \frac{iG}{k}\ln\left(U - \frac{iG}{k}\right)\right)\right\}' - \left(\frac{\hat{w}}{U}\right)'\left(U + \frac{iG}{k}\ln\left(U - \frac{iG}{k}\right)\right). \tag{B 2}$$

Following the usual procedure, we integrate across the discontinuity in U. Assuming that there is no discontinuity in $\hat{w}/U$ (this will be seen to be true below), and that $\hat{w}$ is bounded, we find that

$$[U\hat{w}' - U'\hat{w}] + Ri\left[\frac{\hat{w}}{U}\left(U + \frac{iG}{k}\ln\left(U - \frac{iG}{k}\right)\right)\right] = 0, \tag{B 3}$$

where $[f]$ represents the jump in f across the discontinuity. A second integration (letting one of the limits in (B 3) be a variable) yields the usual non-radiating inviscid condition

$$[\hat{w}/U] = 0, \tag{B 4}$$

which confirms the continuity of $\hat{w}/U$ assumed in deriving (B 3).

Thus, if the numerical integration proceeds to $z = 3-$, the values of $\hat{w}$ and $\hat{w}'$ at $z = 3+$, where boundary condition (3.4) is to be applied, are obtained from (B 3) and (B 4).

Viscous equations

If (2.9b) is integrated twice across the dicontinuity of U and θ, and we assume that there is no discontinuity in $\hat{w}$ (this will be seen true) and that $\hat{w}$ and $\hat{\phi}$ are bounded, then

$$[\hat{\phi}' - P\,Re\,\theta\hat{w}] = 0, \quad [\hat{\phi}] = 0. \tag{B 5a,b}$$

On integrating (2.9a) four times, we find the conditions found by Gage (1972), that

$$[\hat{w}''' - ik\,Re(U\hat{w}' - U'\hat{w})] = 0, \tag{B 6a}$$

$$[\hat{w}'' - ik\,Re\,U\hat{w}] = 0, \tag{B 6b}$$

$$[\hat{w}'] = 0, \quad [\hat{w}] = 0. \tag{B 6c,d}$$

The last condition confirms that there is no discontinuity in $\hat{w}$, as was assumed in arriving at (B 5a).

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